Differential Magnetometry using Singlets

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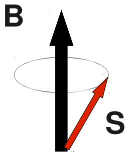
Motivation

- Magnetometry: measuring the magnetic field, $\vec{B}$.

- Example: Spin rotating around the field.



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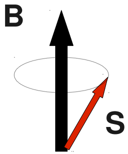

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- Insensitive to homogeneous fields.
- Usually: two clouds. [Wasilewski et. al., Phys. Rev. Lett. 2010; see also Cable, Durkin, Phys. Rev. Lett. 2010.]
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Outline

Multi-Particle Singlet state

- Pure multi-particle singlet states are eigenstates of J_l with 0 eigenvalue, with $J_l = \sum_{n=0}^{N} j_l^{(n)}, \; l = x, y, z$.
- Mixed multi-particle singlet states are mixtures of pure multi-particle singlets.
- They are invariant under the unitary transformations

$$U_{\vec{n}}(\Theta) = \exp\left(-i \frac{J_{\vec{n}}}{\hbar} \Theta \right).$$

- They are in the zero subspace of the Hamiltonian

$$H_{\mathrm{s}} = \kappa(J_x^2 + J_y^2 + J_z^2), \; \kappa > 0,$$

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The singlet state created in spin-squeezing experiments has the following properties:

1. It is permutationally invariant.

$$\varrho = \frac{1}{N!}\sum_{k=1}^{N!}\Pi_k \varrho \Pi_k^\dagger.$$

2. Two ways of expressing it,

$$\varrho_s = \lim_{T\to 0}\frac{e^{-\frac{J_x^2+J_y^2+J_z^2}{T}}}{\mathrm{Tr}\left(e^{-\frac{J_x^2+J_y^2+J_z^2}{T}}\right)}, \text{ and}$$

$$\varrho_s = \sum_{k=1}^{N!}\Pi_k(\left|\Psi^-\right\rangle\left\langle\Psi^-\right| \otimes \cdots \otimes \left|\Psi^-\right\rangle\left\langle\Psi^-\right|)\Pi_k^\dagger,$$

where the two-particle singlet is $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

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Four-particle reduced state

- From the decomposition of the singlet as a product of two-particle singlets, one gets

$$\rho_{1234}^{\rm red} = {\rm Tr}_{56...N}(\varrho_{\rm s}) = \left(\alpha\frac{\mathbb{1}}{16} + \beta\left|\Psi_{12}^{-}\right\rangle\left\langle\Psi_{12}^{-}\right| \otimes \left|\Psi_{34}^{-}\right\rangle\left\langle\Psi_{34}^{-}\right| + \gamma\left|\Psi_{12}^{-}\right\rangle\left\langle\Psi_{12}^{-}\right| \otimes \frac{\mathbb{1}}{2} + \text{permutations}\right).$$

- For β, γ and α we obtain

$$\beta = \frac{(N-4-1)!!}{(N-1)!!} = \frac{1}{(N-1)(N-3)},$$

$$\gamma = \frac{1}{(N-1)} - \frac{1}{(N-1)(N-3)},$$

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- The two-spin reduced density matrix is

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Outline

Geometry of the device

- We consider a singlet of a chain of N particles in the positions

$$(x_n, y_n, z_n) = (nd, 0, 0), \;\; n = 1, ..., N.$$

- The Hamiltonian of the chain in a gradient of magnetic field will be

$$H_{\mathrm{G}} = \gamma B_0 d \sum_{n=1}^{N} n j_z^{(n)} = \hbar\omega_d \sum_{n=1}^{N} n\sigma_z^{(n)}.$$

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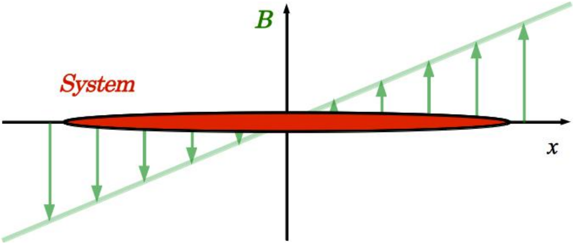

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- We need to compute $\langle J_x^4 \rangle$ in order to obtain $(\Delta\Theta)^2 = \frac{\langle Jx^4 \rangle - \langle Jx^2 \rangle^2}{|\partial_\Theta \langle J_x^2 \rangle|^2}$.

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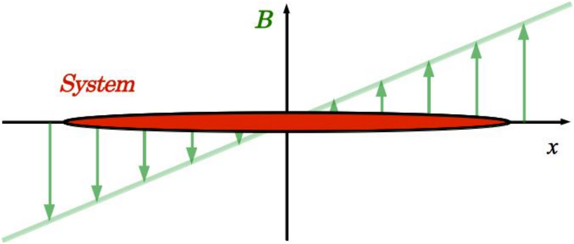

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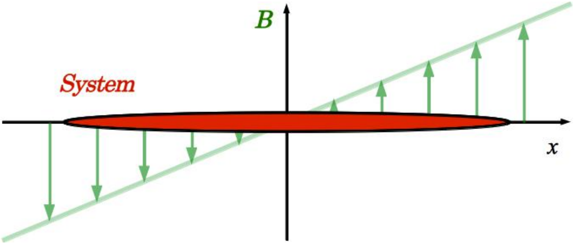

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- We will work in the Heisenberg picture, thus operators will evolve with Θ and expectation values will be computed for the initial state (the singlet state).
- The single-particle operators will evolve on time as

$$j_x^{(n)}(\Theta) = c_n j_x^{(n)} - s_n j_y^{(n)} \equiv X_\Theta^{(n)},$$

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- From the two-body reduced density matrix we know that

$$\langle j_k \otimes j_l \rangle_{\varrho_{12}^{\mathrm{red}}} = -\frac{1}{4(N-1)}\delta_{kl}, \text{ with } k,l = x,y.$$

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- One can get a formula which is easier to compute rewriting the double sum as

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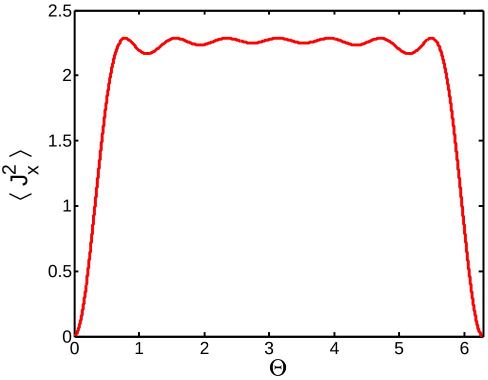
- This is a periodic dynamics with period $T = \frac{2\pi}{\omega_d}$. It grows from 0 to $\sim \frac{N}{4}$ and then it returns to 0.

- Here we plot the dynamics of $\langle J_x^2 \rangle$ as a function of Θ for $N = 8$ particles.



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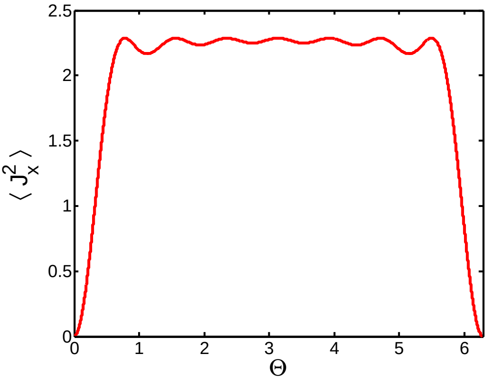

Calculating the accuracy of estimating Θ

- The variance of Θ can be obtained as

$$(\Delta\Theta)^2 = \frac{(\Delta J_x^2)^2}{|\partial_\Theta \langle J_x^2 \rangle|^2}.$$

- For calculating this expression we still need $\langle J_x^4 \rangle$,

$$J_x^4(\Theta) = \left(\sum_n j_x^{(n)}(\Theta) \right)^4.$$

Calculating the accuracy of estimating Θ

- The variance of Θ can be obtained as

$$(\Delta\Theta)^2 = \frac{(\Delta J_x^2)^2}{|\partial_\Theta \langle J_x^2 \rangle|^2}.$$

- For calculating this expression we still need $\langle J_x^4 \rangle$,

$$J_x^4(\Theta) = \left(\sum_n j_x^{(n)}(\Theta) \right)^4.$$

Calculating $\langle J_x^4 \rangle$

- The expectation value of J_x^4 is

$$\langle J_x^4 \rangle(\Theta) = \sum_{n_1,n_2,n_3,n_4} \langle X_\Theta^{(n_1)} X_\Theta^{(n_2)} X_\Theta^{(n_3)} X_\Theta^{(n_4)} \rangle_s.$$

- This can be rewritten as

$$\begin{aligned}
\langle J_x^4 \rangle(\Theta) &= \sum_{n_1} \langle (X_\Theta^{(n_1)})^4 \rangle_s \\
&+ 3 \sum_{n_1 \neq n_2} \langle (X_\Theta^{(n_1)})^2 (X_\Theta^{(n_2)})^2 \rangle_s \\
&+ 4 \sum_{n_1 \neq n_2} \langle (X_\Theta^{(n_1)})^3 (X_\Theta^{(n_2)}) \rangle_s \\
&+ 6 \sum_{\neq(n_1,n_2,n_3)} \langle (X_\Theta^{(n_1)})^2 (X_\Theta^{(n_2)}) (X_\Theta^{(n_3)}) \rangle_s \\
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Calculating $\langle J_x^4 \rangle$

- Again, using $(X_\Theta^{(n)})^2 = \frac{\mathbb{1}}{4}$ we get

$$\begin{aligned}\langle J_x^4 \rangle(\Theta) &= \frac{1}{16}\left[N + 3N(N-1)\right] \\ &+ \left[1 + \frac{3(N-2)}{2}\right] \sum_{n_1 \neq n_2} \langle X_\Theta^{(n_1)} X_\Theta^{(n_2)} \rangle_s \\ &+ \sum_{\neq(n_1,n_2,n_3,n_4)} \langle X_\Theta^{(n_1)} X_\Theta^{(n_2)} X_\Theta^{(n_3)} X_\Theta^{(n_4)} \rangle_s.\end{aligned}$$

- For the four-body correlations we use the reduced four-particle density matrix and we get

$$\begin{aligned}\langle j_x^{(1)} j_x^{(2)} j_x^{(3)} j_x^{(4)} \rangle_{\varrho_{1234}^{\mathrm{red}}} &= \frac{1}{16}\frac{3}{(N-1)(N-3)}, \\ \langle j_y^{(1)} j_y^{(2)} j_y^{(3)} j_y^{(4)} \rangle_{\varrho_{1234}^{\mathrm{red}}} &= \frac{1}{16}\frac{3}{(N-1)(N-3)},\end{aligned}$$

and

$$\langle j_x^{(1)} j_x^{(2)} j_y^{(3)} j_y^{(4)} \rangle_{\varrho_{1234}^{\mathrm{red}}} = \frac{1}{16}\frac{1}{(N-1)(N-3)}.$$

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- Taking everything into account, we come to the expression

$$\begin{aligned}\langle J_x^4 \rangle &= \frac{1}{16}\left[3N^2 - 2N\right] - \frac{3N-4}{8(N-1)}[C^2 + S^2 - N] \\ &+ \frac{3}{16}\frac{1}{(N-1)(N-3)} \sum_{\neq(k,l,m,n)} \Big[c_k c_l c_n c_m \\ &+ s_k s_l s_n s_m + 2 c_k c_l s_n s_m\Big].\end{aligned}$$

- It has $\mathcal{O}(N^4)$ terms.
- Using the same trick that we used for $\langle J_x^2 \rangle$, we simplify the four-index sum as

$$\begin{aligned}&\sum_{\neq(k,l,m,n)} c_k c_l c_n c_m + s_k s_l s_n s_m + 2 c_k c_l s_n s_m \\ &= \sum_{k,l,n,m} c_k c_l c_n c_m + s_k s_l s_n s_m + 2 c_k c_l s_n s_m - P.\end{aligned}$$

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Calculating $\langle J_x^4 \rangle$

- The simplified expression is

$$\begin{aligned}\langle J_x^4\rangle(\Theta) &= \frac{1}{16}\left[3N^2 - 2N\right] + \frac{N(3N-4)}{8(N-1)} \\ &-\frac{3N-4}{8(N-1)}(X_{1,0}^2 + X_{0,1}^2) \\ &+\frac{3}{16}\frac{1}{(N-1)(N-3)}\Big[X_{1,0}^4 + X_{0,1}^4 + 2X_{1,0}^2X_{0,1}^2 \\ &-6X_{4,0} - 6X_{0,4} - 12X_{2,2} + 3X_{2,0}^2 + 3X_{0,2}^2 \\ &+8X_{3,0}X_{1,0} + 8X_{0,3}X_{0,1} + 4X_{1,1}^2 + 8X_{2,1}X_{0,1} \\ &+8X_{1,2}X_{1,0} + 2X_{2,0}X_{0,2} - 6X_{2,0}X_{1,0}^2 - 6X_{0,2}X_{0,1}^2 \\ &-2X_{2,0}X_{0,1}^2 - 2X_{0,2}X_{1,0}^2 - 8X_{1,1}X_{1,0}X_{0,1}\Big].\end{aligned}$$

- Where

$$X_{m,n} = \sum_{p=1}^{N} \cos^m(p\Theta)\sin^n(p\Theta).$$

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Calculating $\langle J_x^4 \rangle$

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& + \frac{3}{16}\frac{1}{(N-1)(N-3)}\left[X_{1,0}^4 + X_{0,1}^4 + 2X_{1,0}^2 X_{0,1}^2\right. \\
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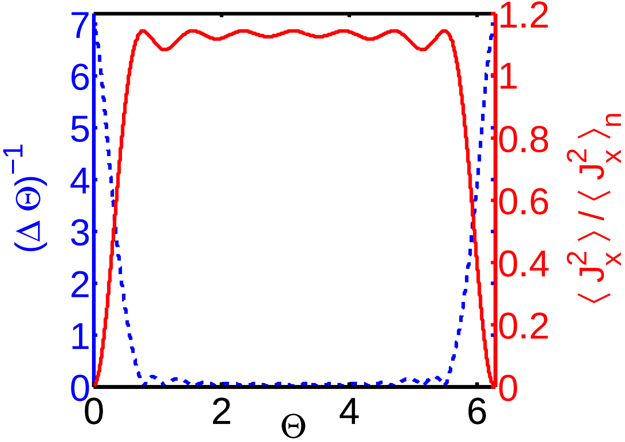
Calculating $\langle J_x^4 \rangle$

- The dynamics of $\langle J_x^2 \rangle$ (solid) and $(\Delta\Theta)^{-2}$ (dashed) as a function of Θ for $N = 8$ particles.



Outline

Arbitrary particle distribution

- Let us consider the case of a one-dimensional continuous density profile.
- It will be characterized by the function $f_N(x_1, ..., x_N)$.
- From it, we define

$$f_{N-p}(x_1, ..., x_{N-p}) = \int \mathrm{d}^p\vec{x} f_N(x_1, ..., x_N),$$

and

$$\hat{f}_M(\alpha_1, ..., \alpha_M) = \int_{\mathbb{R}^M} \mathrm{d}^M\vec{x} f_M(x_1, ..., x_M) e^{i\vec{\alpha}\cdot\vec{x}} = \langle e^{i\vec{\alpha}\cdot\vec{x}} \rangle.$$

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Arbitrary particle distribution

$\langle J_x^2 \rangle$ and $\langle J_x^4 \rangle$ will be

-

$$\begin{aligned}\langle J_x^2 \rangle &= \frac{N}{4} + N(N-1)\int \mathrm{d}x\mathrm{d}y f_2(x,y)\langle X(x,\Theta)X(y,\Theta)\rangle \\ &= \frac{N}{4}\left[1 - \mathrm{Re}\big(\hat{f}_2(\Theta,-\Theta)\big)\right].\end{aligned}$$

-

$$\begin{aligned}\langle J_x^4 \rangle &= \frac{N(3N-2)}{16} \\ &+ \frac{(3N-4)}{2}N(N-1)\int \mathrm{d}x\mathrm{d}y f_2(x,y)\langle X(x,\Theta)X(y,\Theta)\rangle \\ &+ N(N-1)(N-2)(N-3)\int \prod_{i=1}^{4}\mathrm{d}x_i f_4(\{x_i\})\langle \prod_{k=1}^{4} X(x_k,\Theta)\rangle \\ &= \frac{N(3N-2)}{16} - \frac{N(3N-4)}{8}\mathrm{Re}\big(\hat{f}_2(\Theta,-\Theta)\big) \\ &+ \frac{3N(N-2)}{16}\mathrm{Re}\big(\hat{f}_4(\Theta,\Theta,-\Theta,-\Theta)\big).\end{aligned}$$

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Arbitrary particle distribution: particular case of the chain

- For the case of the chain,

$$f_N(x_1, \ldots, x_N) = \frac{1}{N!} \sum_{\sigma \in \Sigma_N} \left(\prod_{k=1}^{N} \delta(x_k - \xi_{\sigma(k)}) \right),$$

and, for $M < N$,

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Arbitrary particle distribution: gaussian cloud

- We will assume that

$$f_M(x_1, ..., x_M) = \prod_{k=1}^{M} f_1(x_k), \ M \leqslant N,$$
$$f_1(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}.$$

- It's Fourier transform is

$$\hat{f}_1(\alpha) = e^{i\alpha x_0} e^{-\frac{\alpha^2\sigma^2}{2}}.$$

- The two and four particle function Fourier transform will be

$$\hat{f}_2(\Theta, -\Theta) = e^{-\sigma^2\Theta^2},$$

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- Finally, we have all the ingredients to determine $\langle J_x^2 \rangle$ and $\langle J_x^4 \rangle$ for the gaussian cloud,

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Outline

Precision of the measurement

- The precision of the measurement will be given by

$$(\Delta\Theta)^{-2} = \frac{|\partial_\Theta \langle J_x^2 \rangle|^2}{\langle J_x^4 \rangle - \langle J_x^2 \rangle}.$$

- For an arbitrary distribution,

$$(\Delta\Theta)^{-2}_{\max} = \lim_{\Theta \to 0} (\Delta\Theta)^{-2} = \frac{N}{L^2}\left[\sigma_x^2 - \mathrm{cov}(x,y)\right].$$

- For the chain, the precision will be

$$(\Delta\Theta)^{-2}_{\max} = \frac{d^2}{L^2}\frac{N^3 + N^2}{12}$$

- For the gaussian cloud, the precision is

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Outline

Conclusions

- We show that multi-particle singlets can be used for differential magnetometry.
- The time dependece of the relevant quantities can be calculated explicitly.
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- FUTURE WORK: Spin-1

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THANK YOU FOR YOUR ATTENTION