Spin squeezing and entanglement in systems of spin-j particles.

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Contents

Definition of spin squeezed state.

- Let's consider an N-particle system and the collective spins

$$J_k := \sum_{n=1}^{N} j_k^{(n)}$$

- One possible definition of squeezing for the J_k is based on the Heisenberg uncertainty principle

$$(\Delta J_x)^2 (\Delta J_y)^2 \geq \frac{1}{4} |\langle J_z \rangle|^2$$

- A state is Spin Squeezed if it has one ”small“ variance

$$(\Delta J_x)^2_{SSS} < (\Delta J_x)^2_{SQL} = \frac{1}{2} |\langle J_z \rangle|$$

where $(\Delta J_x)^2_{SQL} = \frac{1}{2} |\langle J_z \rangle|$ is the Standard Quantum Limit

SSS and metrology

- Usual SSS are completely polarized along the z axis

$$|\langle J_z \rangle| = \frac{N}{2}$$

- and they have a small variance in the x direction

$$(\Delta J_x)^2_{SSS} < \frac{N}{4}$$

- A quantitative definition can be given in terms of a SS parameter

$$\xi_S^2 = \frac{N(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2}$$

$$\xi_S^2(|\Psi\rangle) < 1 \;\Rightarrow\; |\Psi\rangle \text{ is SS}$$

- ξ_S^2 is connected with the precision in sensitivity of phase measurements

$$\xi_S^2 = \frac{(\Delta\phi)^2}{(\Delta\phi)^2_{\mathrm{SQL}}} = \sqrt{N}(\Delta\phi)^2$$

[A. Sørensen, L.M. Duan, J.I. Cirac, and P. Zoller, Nature **409**, 63 (2001); M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993); D.J. Wineland, J. J. Bollinger, and W. M. Itano, Phys. Rev. A **50**, 67 (1994).]

SSS and multiparticle entanglement

- For spin-$1/2$ systems, any separable state, namely of the form

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i^{(1)} \otimes \cdots \otimes \hat{\rho}_i^{(N)}$$

must satisfy

$$\xi_S^2 = \frac{N(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq 1$$

- We will call it Original Spin Squeezing Inequality.
- SS states defined by $\xi_S^2 < 1$ are entangled.
- They can also be useful for quantum information processing protocols.
- For spin-$1/2$ systems SS is directly connected with entanglement.

[A. Sørensen, L.M. Duan, J.I. Cirac, and P. Zoller, Nature **409**, 63 (2001)]

Complete set of SSIs for spin-$1/2$ particles (qubits).

For qubit systems there is also a complete set of (generalized) SSIs

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \le \frac{N(N+2)}{4}$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \ge \frac{N}{2}$$

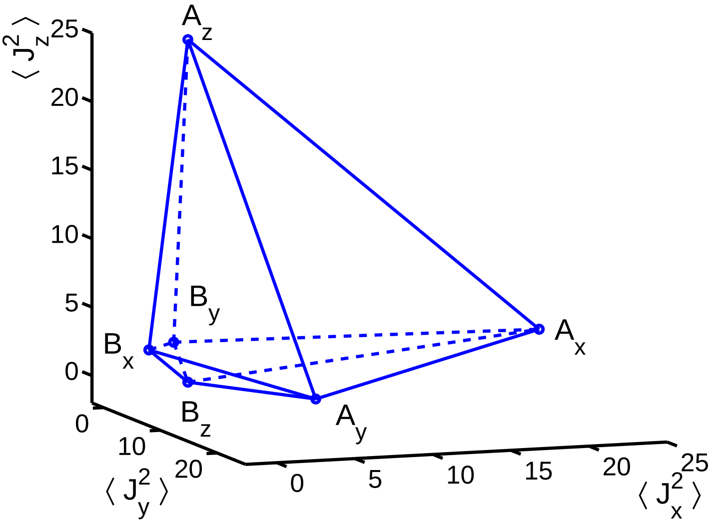
$$(N-1)\left[(\Delta J_x)^2 + (\Delta J_y)^2\right] - \langle J_z^2 \rangle \ge \frac{N(N-2)}{4}$$

$$(N-1)\left[(\Delta J_x)^2\right] - \langle J_y^2 \rangle - \langle J_z^2 \rangle \ge -\frac{N}{2}$$

These are all separability criteria $\Rightarrow$ any state that violates one of them is entangled.

[G. Tóth, C. Knapp, O. Gühne and H.J. Briegel, PRL 99, 250405 (2007)]

The polytope defined by the set of inequalities is filled with separable states.



This is the polytope that results for $\langle \mathbf{J} \rangle = 0$ and $N = 6$.

[G. Tóth, C. Knapp, O. Gühne and H.J. Briegel, PRL 99, 250405 (2007)]

Why consider entanglement connected to SS for higher spin.

- For higher spin systems there are very few entanglement criteria
- Criteria based on collective quantities are easy to use experimentally
- Usually in experiments the particles have many levels (higher spin)
- Some of the levels are screened in order to create artificial 2-level systems (qubits)

- Other experiments for creating entanglement could be easier to prepare
- New interesting entangled states can be prepared
- Different multiparticle systems can be explored

Non entangled SSS for higher spin.

- If we take the same definition of SS for spin-1 particle system

$$\xi_S^2 = \frac{N(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} < 1$$

- then the state

$$|\Phi\rangle = \sqrt{0.6}|1\rangle + \sqrt{1-0.6}|0\rangle$$

- is SS, but it is a $N = 1$ particle state and thus not entangled

Higher spin generalized SSIs.

- Tight entanglement criteria based on $(\Delta J_k)^2, \langle J_k^2 \rangle$ are difficult to obtain
- They can be obtained through numerical optimization

$$(\Delta J_x)^2 + (\Delta J_y)^2 \geq NC_j$$
$$(\Delta J_x)^2 \geq NJF_J(\tfrac{\langle J_z \rangle}{J})$$

[Q. Y. He, S.-G. Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A 84, 022107 (2011); A. Sørensen and K. A. Mølmer Phys. Rev. Lett. 86, 4431 (2001).]

- Or they are useless, because they are not violated

[GV, P. Hyllus, I.L. Egusquiza, and G. Tóth, in preparation.]

The modified second moments $\langle \tilde{J}_k^2 \rangle$.

- We define the modified second moment and variance as

$$\langle \tilde{J}_k^2 \rangle \equiv \langle J_k^2 \rangle - \langle \sum_n (j_k^{(n)})^2 \rangle = \sum_{n \neq m} \langle j_k^{(n)} j_k^{(m)} \rangle,$$

$$\left(\tilde{\Delta} J_k\right)^2 \equiv \langle \tilde{J}_k^2 \rangle - \langle J_k \rangle^2.$$

- They need the measurement of one additional observable

$$\langle M_k \rangle = \langle \sum_n (j_k^{(n)})^2 \rangle$$

- They can be measured as easily as $(\Delta J_k)^2, \langle J_k^2 \rangle$.

Mapping spin-1/2 inequalities to spin-j ones.

We can map spin-1/2 entanglement criteria of the form

$$f(\{\langle J_k \rangle\}, \{\langle \tilde{J}_k^2 \rangle\}) \geq \text{const.},$$

to spin-j ones by doing

$$\langle J_k \rangle \longrightarrow \frac{1}{2j} \langle J_k \rangle, \qquad \langle \tilde{J}_k^2 \rangle \longrightarrow \frac{1}{4j^2} \langle \tilde{J}_k^2 \rangle.$$

Example: Original spin squeezing criterion.

The Original SSI can be mapped to

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} + \frac{Nj^2 - \sum_n \langle (j_x^{(n)})^2 \rangle}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

which is a true separability criterion for spin-j particles.

Complete set of SSIs for spin-j particles.

- We map the complete set of SSIs to

$$\begin{aligned}
\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\leq Nj(Nj+1)\\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq Nj\\
(N-1)(\tilde{\Delta} J_m)^2 - \tilde{J}_k^2 - \tilde{J}_l^2 &\geq -N(N-1)j^2\\
(N-1)\left[(\tilde{\Delta} J_m)^2 + (\tilde{\Delta} J_k)^2\right] - \tilde{J}_l^2 &\geq -N(N-1)j^2
\end{aligned}$$

- Also points saturating the inequalities can be mapped

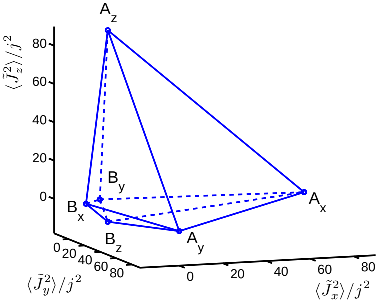
$$\rho_{1/2} = \sum_k p_k \; \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \rho_k^{(3)} \cdots \rightarrow$$

$$\rho_j = \sum_k p_k \; \omega_k^{(1)} \otimes \omega_k^{(2)} \otimes \omega_k^{(3)} \ldots$$

$$|+1/2\rangle_x \rightarrow |+j\rangle_x$$

- Thus the set is complete also for spin-j particle systems

The polytope defined by the set of inequalities is filled with separable states also for spin-j particles.



- This is the polytope that results for $\langle \mathbf{J} \rangle = 0$ and $N = 10$.

The extremal points have coordinates

$$\tilde{\mathbf{K}}_{A_x} := (\langle \tilde{J}_x^2 \rangle, \langle \tilde{J}_y^2 \rangle, \langle \tilde{J}_z^2 \rangle)_{A_x} = 4j^2 \left(\frac{N\kappa}{4} - \kappa(\langle J_y \rangle^2 + \langle J_z \rangle^2), \kappa \langle J_y \rangle^2, \kappa \langle J_z \rangle^2 \right)$$

$$\tilde{\mathbf{K}}_{B_x} = 4j^2 \left(\langle J_x \rangle^2 + \frac{\langle J_y \rangle^2 + \langle J_z \rangle^2}{N}, \kappa \langle J_y \rangle^2, \kappa \langle J_z \rangle^2 \right)$$

where $\kappa = \frac{N-1}{N}$

[GV, P. Hyllus, I.L. Egusquiza, and G. Tóth, in preparation.]

The singlet maximally violates the isotropic inequality.

States that have $\langle J_k \rangle = 0$ and $\langle J_k^2 \rangle = 0$ for all k maximally violate

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Nj$$

Its coordinates in the space of $\tilde{\mathbf{K}}$ are

$$\tilde{\mathbf{K}}_{\mathrm{S}} = \left(-\frac{Nj(j+1)}{3}, -\frac{Nj(j+1)}{3}, -\frac{Nj(j+1)}{3} \right)$$

If we mix a singlet with white noise

$$\rho = (1 - p_n)\rho_S + p_n \frac{\mathbb{1}}{d^N}$$

it is detected as entangled until

$$p_n < \frac{1}{j+1}$$

i.e. the noise tolerance decrease with j.

The symmetric Dicke states violates the single variance inequality.

The inequality with only one variance

$$(N-1)(\tilde{\Delta}J_z)^2 - \tilde{J}_x^2 - \tilde{J}_y^2 \geq -N(N-1)j^2$$

is maximally violated by symmetric Dicke states. They have coordinates

$$\tilde{\mathbf{K}}_{\mathrm{Dicke}} = \left(\frac{N^2j^2}{2}, \frac{N^2j^2}{2}, -Nj^2\right)$$

If we mix it with white noise

$$\rho = (1-p_n)\rho_{\mathrm{Dicke}} + p_n\frac{\mathbb{1}}{d^N}$$

it is detected as entangled until:

$$p_n < \frac{N}{2N-1}$$

i.e. the noise tolerance goes to $1/2$ for large N independently of j.

The single variance inequality is strictly stronger than the original SSI.

- The single variance inequality

$$(N-1)(\tilde{\Delta} J_x)^2 - \tilde{J}_y^2 - \tilde{J}_z^2 \geq -N(N-1)j^2 \qquad (1)$$

  is strictly stronger than the mapped original SSI

$$\frac{(\tilde{\Delta} J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} + \frac{Nj^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}. \qquad (2)$$

- In particular Eq. (1) detects Dicke states

$$\tilde{\mathbf{K}}_{\mathrm{Dicke}} = \left(\frac{N^2 j^2}{2}, \frac{N^2 j^2}{2}, -Nj^2\right)$$

  while Eq. (2) does not, since

$$\mathbf{J}_{\mathrm{Dicke}} = (0, 0, 0)$$

- The complete set of SSIs also detects unpolarized states.

Expression of the SSIs for completely polarized ensembles.

- For ensembles of particles in which $\langle J_z \rangle \simeq Nj$ we define

$$x = \frac{J_x}{\sqrt{Nj}} \quad \text{and} \quad y = \frac{J_y}{\sqrt{Nj}}$$

- Then, the two inequalities

$$\frac{(\tilde{\Delta} J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} + \frac{Nj^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}$$
$$(N-1)(\tilde{\Delta} J_x)^2 - \tilde{J}_y^2 - \tilde{J}_z^2 \geq -N(N-1)j^2$$

- both reduce to the simple condition

$$(\Delta x)^2 \geq j$$

The set of generalized SSIs

In the plane $\left((\tilde{\Delta}J_z)^2, \langle \tilde{J}_x^2 + \tilde{J}_y^2 \rangle\right)$ the complete set of SSI is

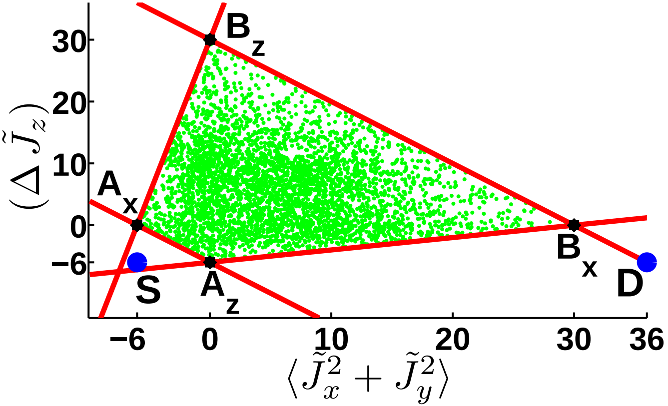


Figure: It is depicted the polytope defined by the SSIs. The interior points are random separable states. The two exterior points are the singlet and the Dicke state.

Experimental measurements of $\langle \tilde{J}_k^2 \rangle$.

- In order to measure $\langle \tilde{J}_k^2 \rangle$ one needs

$$\langle J_k^2 \rangle \quad \text{and} \quad \langle M_k \rangle = \langle \sum_n (j_k^{(n)})^2 \rangle$$

- They can be actually measured in many experimental systems, like BECs and atomic ensembles.

- $\langle M_k \rangle$ can be obtained from the populations n_{j_k} of the eigenstates of j_k.
- In the practice only one $\langle M_k \rangle$ has to be measured to evaluate the SSIs.

Conclusions

- We found high spin generalizations of the complete set of SSIs.
- In particular, also the completeness of the set is extended.
- We have studied also states that saturate or violate those inequalities.
- Compared with usual SSI the complete set also detects unpolarized states, that are very useful to produce.
- The SS parameters coming from the complete set are able to detect many interesting entangled states.
- They just need the measurement of a few collective quantities.

THANKS FOR YOUR ATTENTION!

[GV, P. Hyllus, I.L. Egusquiza, and G. Tóth, Phys. Rev. Lett. **107**, 240502 (2011); GV, P. Hyllus, I.L. Egusquiza, and G. Tóth, in preparation]

The set of SSIs in terms of ρ_{av2}.

- Defining the average 2-particle reduced density matrix as

$$\rho_{\mathrm{av2}} := \frac{1}{N(N-1)} \sum_{i \neq j} \rho_{ij},$$

- we can put the complete set of SSIs in the (compact) form

$$\Sigma \leq j^2 - N \sum_{k \in I} (\langle j_k \rangle_{\mathrm{av2}}^2 - \langle j_k \otimes j_k \rangle_{\mathrm{av2}})$$

- where

$$\Sigma := \sum_{k=x,y,z} \langle j_k \otimes j_k \rangle_{\mathrm{av2}}$$

- and we have to consider the possibilities
  $I = \{\emptyset\}, \{x\}, \{x,y\}, \{x,y,z\}$

Extremal states of the complete set of SSIs.

- The vertices of the polytope are states of the forms

$$\rho_{\rm prod} = \rho_+^{\otimes M} \otimes \rho_-^{\otimes (N-M)},$$
$$\rho_{\rm symm} = p\rho_+^{\otimes N} + (1-p)\rho_-^{\otimes N},$$

where the single particle states are $\rho_+ = |j\rangle\langle j|$ and $\rho_- = |-j\rangle\langle -j|$

- They are suitable as initial state in order to create spin squeezing

- For example in atomic ensembles systems.

[G. Tóth and M. Mitchell, New J. Phys. **12**, 053007 (2010).]

The planar squeezed state violates the two variances inequality.

- Planar squeezed states satisfy

$$(\Delta J_x)^2 + (\Delta J_y)^2 = C_J \sim J^{2/3}$$

[Q. Y. He, S.-G. Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A 84, 022107 (2011).]

- They also violate the inequality with two variances

$$(N-1)\left[(\tilde{\Delta} J_x)^2 + (\tilde{\Delta} J_y)^2\right] - \tilde{J}_z^2 \geq -N(N-1)j^2$$

With this inequality also imperfect planar squeezed states can be detected as entangled.

Coordinate independent form of SSIs

Defining the matrices

$$
\begin{aligned}
C_{kl} &= \frac{1}{2}\langle J_k J_l + J_l J_k \rangle, \\
\gamma_{kl} &= C_{kl} - \langle J_k \rangle \langle J_l \rangle, \\
D_{kl} &= \frac{1}{2}\langle \sum_m j_k^{(m)} j_l^{(m)} + j_l^{(m)} j_k^{(m)} \rangle, \\
\Xi &= (N-1)\gamma + C - ND,
\end{aligned}
$$

we can put the complete set of SSIs in two equivalent compact, coordinate independent forms

$$
\begin{aligned}
(N-1)\mathrm{Tr}\,(\gamma) &\geq \sum_{n=1}^{3-|I|} \lambda_n^{\downarrow}(\Xi) + N(N-1)j, \\
\mathrm{Tr}\,(C) &\leq \sum_{n=1}^{|I|} \lambda_n^{\uparrow}(\Xi) + Nj(Nj+1).
\end{aligned}
$$

where we have to consider the possibilities $|I| = 0, 1, 2, 3$.

Generalized spin squeezing parameter

- We define a spin squeezing parameter in two different form

$$\xi_G = \min_{|I|=1,2,3} \frac{\sum_{n=1}^{|I|} \lambda_n^{\uparrow}(\Xi) + Nj(Nj+1)}{\mathrm{Tr}\,(C)},$$

$$\tau_G = \min_{|I|=1,2,3} \frac{(N-1)\mathrm{Tr}\,(\gamma) + Nj(Nj+1)}{\sum_{n=1}^{3-|I|} \lambda_n^{\downarrow}(\Xi) + N^2 j(j+1)}.$$

- All CSS satisfy $\xi_G = \tau_G = 1$ and all separable states satisfy

$$\xi_G \geq 1 \quad \text{and} \quad \tau_G \geq 1$$

- All possible entangled spin squeezed states can be detected with $\xi_G < 1$, $\tau_G < 1$, including e.g. the singlet and the symmetric Dicke states.

Iterative computation of ξ_G, τ_G.

- Since the eigenvalues $\lambda_n(\Xi)$ can be also negative there is a simple procedure to look directly at the minima

$$\xi_G = \min_{|I|=1,2,3} \frac{\sum_{n=1}^{|I|} \lambda_n^{\uparrow}(\Xi) + Nj(Nj+1)}{\mathrm{Tr}\,(C)},$$

$$\tau_G = \min_{|I|=1,2,3} \frac{(N-1)\mathrm{Tr}\,(\gamma) + Nj(Nj+1)}{\sum_{n=1}^{3-|I|} \lambda_n^{\downarrow}(\Xi) + N^2 j(j+1)},$$

- 1. for ξ_G one finds iteratively the smallest eigenvalues until one gets a positive value
  2. then take the sum of just the negative ones.

- 1. for τ_G one finds iteratively the biggest eigenvalues until one gets a negative value
  2. then take the sum of just the positive ones.