Optimal inequalities for state-independent contextuality

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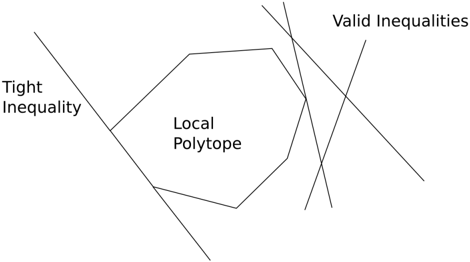
Local hidden variable model: result of a measurement in a space-time region is independent of which measurement is performed in another space-like separated region.

Bell inequalities are constraints on correlations between the results of spacelike-separated measurements, satisfied by any LHV theory and violated by QM.
Tight Bell inequalities correspond to facets of the local polytope, i.e., $(d-1)$-dimensional faces of the d-dimensional polytope obtained as convex hull of vectors representing local assignments to measurements.
Properties:

- Minimal set of inequalities (finite set)
- Necessary and sufficient conditions for LHV
- Boundaries between classical and quantum correlations



Vector representing probability assignments
$p = (p_1, \ldots, p_n, \ldots p_{ij}, \ldots, p_{i_1 \ldots i_m}, \ldots)$
convex combination of deterministic assignments
$u_\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n, \ldots, \varepsilon_i \varepsilon_j, \ldots, \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_m}, \ldots), \ \varepsilon_i \in \{0, 1\}$
i.e. $p = \sum_\varepsilon \lambda_\varepsilon u_\varepsilon$, with $\lambda_\varepsilon \geq 0$ and $\sum_\varepsilon \lambda_\varepsilon = 1$

Noncontextual hidden variable model: result of a measurement is independent of which (compatible) measurement is performed jointly.

Noncontextuality (NC) inequalities are constraints on correlations among results of compatible observables, satisfied by any noncontextual HV model.
Analogous notion of tightness (w.r.t. Bell inequalities): mathematically identical, but based on a different (more general) notion of *joint measurability*.
As opposed to Bell inequalities, NC inequalities

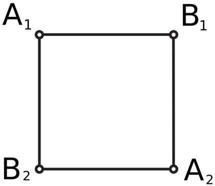
- Do not need composite systems
- Do not need entangled states
- May reveal state-independent contextuality (SIC), i.e. state-independent violation[1]

[1] Cabello PRL **101** (2008), Badziąg, Bengtsson, Cabello, Pitowsky, PRL **103** (2009)

Some examples:
Tight noncontextuality inequalities

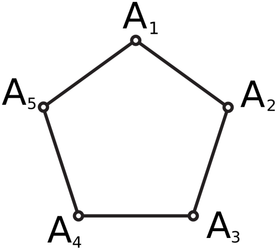
- CHSH inequality

$$\langle A_1 B_1\rangle+\langle A_2 B_1\rangle+\langle A_2 B_2\rangle-\langle A_1 B_2\rangle \leq 2$$



- Klyachko, Can, Binicioğlu and Shumovsky (KCBS)[2]

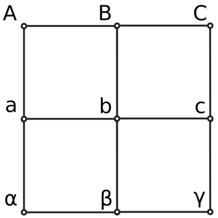
$$\langle A_1 A_2\rangle + \langle A_2 A_3\rangle + \langle A_3 A_4\rangle + \langle A_4 A_5\rangle - \langle A_1 A_5\rangle \leq 3$$



[2] Klyachko *et al.* PRL **101** (2008)

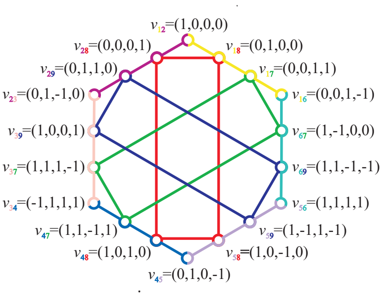
Tight SIC noncontextuality inequalities[3]

- Peres-Mermin square inequality



$$\langle ABC\rangle + \langle abc\rangle + \langle \alpha\beta\gamma\rangle + \langle Aa\alpha\rangle + \langle Bb\beta\rangle - \langle Cc\gamma\rangle \leq 4$$

- 18 vector KS proof



$$-\langle A_{12}A_{16}A_{17}A_{18}\rangle - \langle A_{12}A_{23}A_{28}A_{29}\rangle - \langle A_{23}A_{34}A_{37}A_{39}\rangle - \langle A_{34}A_{45}A_{47}A_{48}\rangle - \langle A_{45}A_{56}A_{58}A_{59}\rangle - \langle A_{16}A_{56}A_{67}A_{69}\rangle - \langle A_{17}A_{37}A_{47}A_{67}\rangle - \langle A_{18}A_{28}A_{48}A_{58}\rangle - \langle A_{29}A_{39}A_{59}A_{69}\rangle \leq 7.$$

[3] Cabello PRL **101** (2008)

Finding optimal inequalities, a general approach

- Obtaining all tight inequalities: Hard task[4].
- There exist algorithms, but the time required grows enormously with the number of measurement settings. Method applied only in simple cases.
- Given the facets, one has to find quantum observables and state that maximize the gap between classical and quantum values.
- SIC?

[4] Pitowsky, *Quantum probability, Quantum Logic (1989)*

In many cases quantum observables are already known (e.g., proofs of Kochen-Specker theorem)

- SIC.
- Maximize quantum state-independent violation via linear programming (solved with standard optimization routines, optimality of the solution guaranteed).
- Find tight inequalities among optimal solutions.
- Additional properties (e.g., vanishing coefficients simplify experimental test of inequality).

Some notation:

$A_1, \ldots, A_n$ dichotomic observables,

Context $\underline{\mathfrak{c}}$: set of indices s.t. $[A_k, A_l] = 0$ for all $k, l \in \underline{\mathfrak{c}}$,

Contextuality scenario: $\mathfrak{C} = \{\underline{\mathfrak{c}}\}$,

Noncontextual assignments for observables: $\vec{a} = (a_1, \ldots, a_n) \in \{-1, 1\}^n$,

Correlation vector: $\vec{v} = (v_{\underline{\mathfrak{c}}} = \langle \prod_{k \in \underline{\mathfrak{c}}} a_k \rangle \mid \underline{\mathfrak{c}} \in \mathfrak{C})$,

General form for a noncontextuality inequality :

$$\sum_{\underline{\mathfrak{c}} \in \mathfrak{C}} \lambda_{\underline{\mathfrak{c}}} \langle \prod_{k \in \underline{\mathfrak{c}}} a_k \rangle = \vec{\lambda} \cdot \vec{v} \leq \eta$$

with η classical bound.

In QM to each $v_{\underline{\mathfrak{c}}}$ corresponds $\langle \prod_{k \in \underline{\mathfrak{c}}} A_k \rangle_\rho$, with ρ quantum state

Maximal violation given by $[\max_\rho \langle T(\vec{\lambda}) \rangle_\rho]/\eta$, with

$$T(\vec{\lambda}) = \sum_{\underline{\mathfrak{c}} \in \mathfrak{C}} \lambda_{\underline{\mathfrak{c}}} \prod_{k \in \underline{\mathfrak{c}}} A_k$$

SIC allows to efficently find the optimal inequalities via the LP:

$$\begin{aligned} \text{minimize:} \quad & \eta, \\ \text{subject to:} \quad & T(\vec{\lambda}) = \mathbb{1}, \text{ and} \\ & \sum_{\underline{\mathfrak{c}} \in \mathfrak{C}} \lambda_{\underline{\mathfrak{c}}} \prod_{k \in \underline{\mathfrak{c}}} a_k \leq \eta \text{ for all assignments } \vec{a}, \end{aligned}$$

Implemented with CVXOPT for Python.
Contextuality scenario exhibits SIC $\Leftrightarrow \eta < 1$ solution.
More general: semidefinite program obtained by substituting $T(\vec{\lambda}) = \mathbb{1}$ with $T(\vec{\lambda}) - \mathbb{1} \geq 0$.
(For the cases we considered linear and semidefinite program yield the same result).

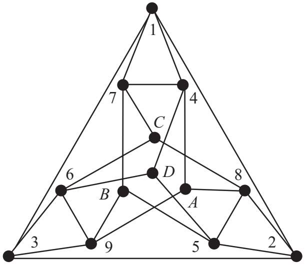
Optimal solution $\vec{\lambda}^*$ not unique, solutions form a convex polytope defined by the inequalities

$$\sum_{\underline{\mathfrak{c}}\in\mathfrak{C}} \lambda_{\underline{\mathfrak{c}}} \prod_{k\in\underline{\mathfrak{c}}} a_k \leq \eta^*$$

for any assignment $\vec{a}$, with η^* optimal value.
In such a convex set we look for additional properties:

- tightness ($\Rightarrow$ extremal point of the set of solutions)
- vanishing coefficients (less correlations to be tested experimentally)

Yu and Oh's scenario[5]: observables $A_i = \mathbb{1} - 2|v_i\rangle\langle v_i|$ defined by



v_1=(1,0,0) v_5=(1,0,-1) v_A=(-1,1,1)
v_2=(0,1,0) v_6=(1,-1,0) v_B=(1,-1,1)
v_3=(0,0,1) v_7=(0,1,1) v_C=(1,1,-1)
v_4=(0,1,-1) v_8=(1,0,1) v_D=(1,1,1)
v_9=(1,1,0)

$\underline{\mathfrak{c}}$	YO	opt$_2$	opt$_3$	$\underline{\mathfrak{c}}$	YO	opt$_2$	opt$_3$	$\underline{\mathfrak{c}}$	YO	opt$_2$	opt$_3$
1	2	2	1	A–D	4	2	2	3, 9	−1	−2	−1
2	2	3	1	1,2	−1	−1	−2	4, 7	−1	**0**	−1
3	2	3	1	1,3	−1	−1	−2	5, 8	−1	−2	−1
4	2	1	1	1,4	−1	−1	−1	6, 9	−1	−2	−1
5	2	2	1	1,7	−1	−1	−1	∗, A–D	−1	**−1**	−2
6	2	2	1	2,3	−1	−2	−2	1,2,3	–	–	**0**
7	2	1	1	2,5	−1	−2	−1	1,4,7	–	–	−3
8	2	2	1	2,8	−1	−2	−1	2,5,8	–	–	−3
9	2	2	1	3,6	−1	−2	−1	3,6,9	–	–	−3

TABLE I: Coefficients $\lambda_{\underline{\mathfrak{c}}}$ of inequalities for the Yu & Oh scenario. The column $\underline{\mathfrak{c}}$ labels the different contexts, YO the original coefficients, opt$_2$ an optimal tight inequality with contexts of maximal size 2, opt$_3$ an optimal tight inequality with contexts of all sizes. The coefficients in the column YO have been multiplied by 50/3, for the column opt$_2$ by 52/3 and for the column opt$_3$ by 83/3.

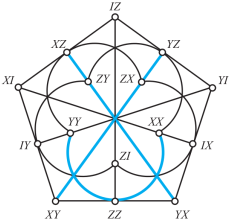
Quantum violations[6]$(1/\eta)$: $\eta_{YO}^{-1} \approx 1.042$, $\eta_2^{-1} \approx 1.083$, $\eta_3^{-1} \approx 1.107$

[6]Yu, Oh PRL **108** (2012)
[6]Cabello, Budroni, Gühne, Kleinmann, Larsson arXiv:1204.3741

Further examples:

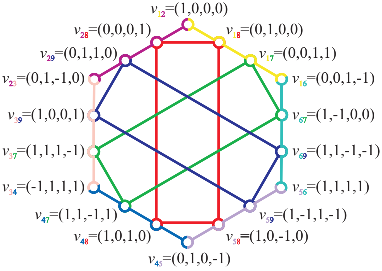
Extended Peres-Mermin square:

- Optimal violation: $1/\eta = 5/3$
- Obtained with coefficients $\lambda = 1/9$ for black lines and $\lambda = -1/9$ for blue lines.



18 vector KS proof:

- For contexts up to size 2: $1/\eta = 18/17 \approx 1.059$
- Size 3: $1/\eta = 8/7 \approx 1.143$
- Size 4: $1/\eta = 9/7 \approx 1.286$



Conclusions

- State-independent contextuality condition allows for optimization.
- Several tight inequalities are among the optimal solutions.
- Additional properties can be imposed (e.g., vanishing coefficients).
- We applied our method to various SIC scenarios maximizing the quantum violation or showing that known inequalities are optimal.